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Resolvents of operators and partial functional differential equations with nonautonomous past

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Abstract

To a backward evolution family $\mathcal{U} = (U(t, s))_{t \leq s \leq 0}$ on a Banach space X we associate an abstract differential operator G through the integral equation $u(t) = U(t, s)u(s) + \int_t^s U(t, \xi)f(\xi)d\xi$ on a Banach space of X -valued functions on \mathbb{R}_- . We compute the resolvent of the restriction of this operator to a smaller domain to obtain a generator. We then apply the results to prove existence, exponential stability and exponential dichotomy of solutions to partial functional equations with nonautonomous past as discussed in [S. Brendle, R. Nagel, Dist. Contin. Dynam. Systems 8 (2002) 953–966]. Our main tools are spectral mapping theorems for evolution semigroups and hyperbolicity criteria.

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1. Introduction

Motivated, e.g., by delay equations with diffusion, Brendle and Nagel studied in [1] the following system of equations:

$$\frac{\partial}{\partial t}u(t, 0) = Bu(t, 0) + \Phi u(t, \cdot), \quad t \geq 0, \quad (1.1)$$

$$\frac{\partial}{\partial t}u(t, s) = \frac{\partial}{\partial s}u(t, s) + A(s)u(t, s), \quad t \geq 0 \geq s. \quad (1.2)$$

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Here, the function $u(\cdot, \cdot)$ takes values in a Banach space X , B is a linear operator on X , and Φ , called the *delay operator*, is a linear operator from a space of X -valued functions on \mathbb{R}_- into X . Finally, $A(s)$ are (unbounded) operators on X for which the nonautonomous Cauchy problem

$$\begin{cases} \frac{dx(t)}{dt} = -A(t)x(t), & t \leq s \leq 0, \\ x(s) = x_s \in X \end{cases} \quad (1.3)$$

is well-posed with exponential bound. In particular, there exists an exponentially bounded backward evolution family $\mathcal{U} = (U(t, s))_{t \leq s \leq 0}$ solving (1.3), i.e., the solutions of (1.3) are given by $x(t) = U(t, s)x(s)$ for $t \leq s \leq 0$.

These equations describe a system with delay (Eq. (1.1)) acting on a nonautonomous past (Eq. (1.2)) and has been solved using semigroup methods in the space $C_0(\mathbb{R}_-, X)$ in [1] or in the space $L^p(\mathbb{R}_-, X)$ in [4].

In this paper we use the theory of evolution semigroups as developed by Chicone and Latushkin [2], Schnaubelt ([3, Chap. VI.9] and [18]) and others (see [11, 13]) to define an abstract differential operator G on $C_0(\mathbb{R}_-, X)$ (see Definition 2.4). We then use the delay operator Φ (and the operator B) to define a restriction $G_{B, \Phi}$ of G . For this restriction we compute explicitly its resolvent and show the Hille–Yosida estimates. In this way, we obtain a semigroup $(T_{B, \Phi}(t))_{t \geq 0}$ which solves (1.1) and (1.2) in a mild sense (see [1, Sections 1 and 2]). The advantage of our method, using direct descriptions of resolvents of generators, is that it yields explicit stability estimates. In particular, we can show that the exponential stability and exponential dichotomy of this semigroup, hence of the solutions of (1.1) and (1.2), is robust under small perturbations of the delay operator Φ .

2. Evolution semigroups

In this section we start from an evolution family \mathcal{U} on \mathbb{R}_- and extend it to all of \mathbb{R} in order to define a corresponding evolution semigroup on $C_0(\mathbb{R}, X)$. For most of these concepts we refer to the monograph by Chicone and Latushkin [2] or the survey articles by Schnaubelt ([18] or [3, Chap. VI.9]).

Definition 2.1. A family of operators $\mathcal{U} = (U(t, s))_{t \leq s \leq 0}$ on a Banach space X is called a (*strongly continuous, exponentially bounded*) *backward evolution family* on \mathbb{R}_- if

- (i) $U(t, t) = \text{Id}$ and $U(t, r)U(r, s) = U(t, s)$ for $t \leq r \leq s \leq 0$,
- (ii) the map $(t, s) \mapsto U(t, s)x$ is continuous for every $x \in X$,
- (iii) there are constants $N \geq 1$ and $\omega_1 \in \mathbb{R}$ such that $\|U(t, s)\| \leq Ne^{\omega_1(s-t)}$ for $t \leq s \leq 0$.

The constant

$$\omega(\mathcal{U}) := \inf\{\alpha \in \mathbb{R}: \exists H \geq 1 \text{ such that } \|U(t, s)\| \leq He^{\alpha(s-t)} \text{ for all } t \leq s \leq 0\}$$

is called the *growth bound* of \mathcal{U} .

In order to define a corresponding evolution semigroup (see, e.g., [2,11], or [3, Chap. VI.9]) we first extend $(U(t, s))_{t \leq s \leq 0}$ to a backward evolution family $(\tilde{U}(t, s))_{t \leq s}$ on \mathbb{R} . This can be done by setting

$$\tilde{U}(t, s) := \begin{cases} U(t, s) & \text{for } t \leq s \leq 0, \\ U(t, 0) & \text{for } t \leq 0 \leq s, \\ U(0, 0) = \text{Id} & \text{for } 0 \leq t \leq s. \end{cases}$$

Definition 2.2. On $\tilde{E} := C_0(\mathbb{R}, X)$, the evolution semigroup $(\tilde{T}(t))_{t \geq 0}$ corresponding to $(\tilde{U}(t, s))_{t \leq s}$ is given by

$$(\tilde{T}(t)\tilde{f})(s) := \tilde{U}(s, s+t)\tilde{f}(s+t) = \begin{cases} U(s, s+t)\tilde{f}(s+t) & \text{for } s \leq s+t \leq 0, \\ U(s, 0)\tilde{f}(s+t) & \text{for } s \leq 0 \leq s+t, \\ \tilde{f}(s+t) & \text{for } 0 \leq s \leq s+t, \end{cases}$$

for all $\tilde{f} \in \tilde{E}$, $s \in \mathbb{R}$, $t \geq 0$.

One can prove that this semigroup is strongly continuous on \tilde{E} (see [3, Lemma VI.9.10]). We denote its generator by $(\tilde{G}, D(\tilde{G}))$. The following properties of this operator can be shown as in [10, Lemma 1] and [15, Theorem 2.4].

Lemma 2.3. For \tilde{u}, \tilde{f} in \tilde{E} and $\lambda \in \mathbb{C}$ the following assertions hold:

- (i) $\tilde{u} \in D(\tilde{G})$ and $(\lambda - \tilde{G})\tilde{u} = \tilde{f}$ if and only if \tilde{u} and \tilde{f} satisfy the integral equation

$$\tilde{u}(t) = e^{\lambda(t-s)}\tilde{U}(t, s)\tilde{u}(s) + \int_t^s e^{\lambda(t-\xi)}\tilde{U}(t, \xi)\tilde{f}(\xi) d\xi \quad \text{for } t \leq s. \quad (2.1)$$

- (ii) The operator $(\tilde{G}, D(\tilde{G}))$ is a local operator in the sense that for $\tilde{u} \in D(\tilde{G})$ and $\tilde{u}(s) = 0$ for all $a < s < b$ we have that $[\tilde{G}\tilde{u}](s) = 0$ for all $a < s < b$.

The locality of \tilde{G} allows us to define an operator G on $E := C_0(\mathbb{R}_-, X)$.

Definition 2.4. Take

$$D(G) := \{\tilde{f}|_{\mathbb{R}_-} : \tilde{f} \in D(\tilde{G})\}$$

and define

$$[Gf](t) = [\tilde{G}\tilde{f}](t) \quad \text{for } t \leq 0 \quad \text{and} \quad f = \tilde{f}|_{\mathbb{R}_-}.$$

Analogously to Lemma 2.3 we have the following description of G .

Lemma 2.5. Let $u, f \in E$ and $\lambda \in \mathbb{C}$. Then $u \in D(G)$ and $(\lambda - G)u = f$ if and only if u and f satisfy

$$u(t) = e^{\lambda(t-s)}U(t, s)u(s) + \int_t^s e^{\lambda(t-\xi)}U(t, \xi)f(\xi) d\xi \quad \text{for } t \leq s \leq 0. \quad (2.2)$$

Proof. If $u, f \in E$ satisfy Eq. (2.2), then we extend u, f to the whole line by

$$\begin{aligned}\tilde{u}(t) &:= \begin{cases} u(t) & \text{for } t \leq 0, \\ e^{\lambda t} g(t) & \text{for } t > 0, \end{cases} \\ \tilde{f}(t) &:= \begin{cases} f(t) & \text{for } t \leq 0, \\ -e^{\lambda t} g'(t) & \text{for } t > 0. \end{cases}\end{aligned}$$

Here, $g : \mathbb{R}_+ \rightarrow X$ is continuously differentiable with compact support such that $g(0) = u(0)$, $g'(0) = -f(0)$. Then \tilde{u}, \tilde{f} belong to $\tilde{E} = C_0(\mathbb{R}, X)$. A straightforward computation yields that \tilde{u} and \tilde{f} satisfy Eq. (2.1). Hence, by Lemma 2.3, we obtain that the equality $(\lambda - \tilde{G})\tilde{u} = \tilde{f}$ holds. By definition of G we have that $u \in D(G)$ and $(\lambda - G)u = f$.

Conversely, if $u \in D(G)$ and $(\lambda - G)u = f$, then, by the definition of G , there exist $\tilde{u}, \tilde{f} \in C_0(\mathbb{R}, X)$ such that $\tilde{u}|_{\mathbb{R}_-} = u$, $\tilde{f}|_{\mathbb{R}_-} = f$ and $(\lambda - \tilde{G})\tilde{u} = \tilde{f}$. By Lemma 2.3, \tilde{u} and \tilde{f} satisfy Eq. (2.1). Restricting this equation to \mathbb{R}_- we have that u, f satisfy (2.2). \square

We note that such an operator G has been used to study the asymptotic behavior of evolution families on the half-line (see [7,11,12]). The operator G becomes a generator only if we restrict it to a smaller domain, e.g., $D := \{u \in D(G) : [Gu](0) = 0\}$ (see [11, Lemma 1.1]). However, for later applications we consider a more general case and make the following assumption.

Assumption 2.6. Let $(B, D(B))$ be the generator of a strongly continuous semigroup $(e^{tB})_{t \geq 0}$ on the Banach space X satisfying $\|e^{tB}\| \leq Me^{\omega_2 t}$ for some constants $M \geq 1$ and $\omega_2 \in \mathbb{R}$.

Definition 2.7. On the space E we define an evolution semigroup $(T_{B,0}(t))_{t \geq 0}$ by

$$[T_{B,0}(t)f](s) = \begin{cases} U(s, s+t)f(s+t) & \text{for } s+t \leq 0, \\ U(s, 0)e^{(t+s)B}f(0) & \text{for } s+t \geq 0, \end{cases} \quad \text{for all } f \in E.$$

One can easily verify that $(T_{B,0}(t))_{t \geq 0}$ is strongly continuous. We denote its generator by $G_{B,0}$.

We then have the following properties of $G_{B,0}$ and $(T_{B,0}(t))_{t \geq 0}$.

Proposition 2.8. *The following assertions hold:*

(i) *The generator of $(T_{B,0}(t))_{t \geq 0}$ is given by*

$$\begin{aligned}D(G_{B,0}) &:= \{f \in D(G) : f(0) \in D(B) \text{ and } (G(f))(0) = Bf(0)\}, \\ G_{B,0}f &:= Gf \quad \text{for } f \in D(G_{B,0}).\end{aligned}$$

(ii) *The set $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega(\mathcal{U}) \text{ and } \lambda \in \rho(B)\}$ is contained in $\rho(G_{B,0})$. Moreover, for λ in this set, the resolvent is given by*

$$\begin{aligned}[R(\lambda, G_{B,0})f](t) &= e^{\lambda t} U(t, 0)R(\lambda, B)f(0) \\ &\quad + \int_t^0 e^{\lambda(t-\xi)} U(t, \xi)f(\xi) d\xi \quad \text{for } f \in E, \quad t \leq 0.\end{aligned}$$

(iii) The semigroup $(T_{B,0}(t))_{t \geq 0}$ satisfies

$$\|T_{B,0}(t)\| \leq K e^{\omega t}$$

with $K := MN$ and $\omega := \max\{\omega_1, \omega_2\}$ for the constants M, N, ω_1 and ω_2 appearing in Definition 2.1 and Assumption 2.6.

Proof. (i) This can be found in [1, Proposition 2.8].

(ii) Observe that for $f \in E$, $\lambda \in \rho(B)$ and $\operatorname{Re} \lambda > \omega(\mathcal{U})$ the function

$$u(t) := e^{\lambda t} U(t, 0) R(\lambda, B) f(0) + \int_t^0 e^{\lambda(t-\xi)} U(t, \xi) f(\xi) d\xi, \quad t \leq 0,$$

belongs to E and is the unique solution of Eq. (2.2) with the initial condition $u(0) = R(\lambda, B) f(0)$. This condition is equivalent to $(\lambda - B)u(0) = f(0) = [(\lambda - G)u](0)$ or $[Gu](0) = Bu(0)$. This means that $u \in D(G_{B,0})$ and $u = R(\lambda, G_{B,0})f$.

(iii) This follows immediately from the definition of $(T_{B,0}(t))_{t \geq 0}$. \square

3. Evolution semigroups with bounded delay

In this section we shall consider a bounded linear operator $\Phi : E \rightarrow X$, called *delay operator*, and use it to define the following restriction of the operator G from Definition 2.4.

Definition 3.1. The operator $G_{B,\Phi}$, $D(G_{B,\Phi})$ on E is given by

$$\begin{aligned} D(G_{B,\Phi}) &:= \{f \in D(G) : f(0) \in D(B) \text{ and } (Gf)(0) = Bf(0) + \Phi f\}, \\ G_{B,\Phi} f &:= Gf \quad \text{for } f \in D(G_{B,\Phi}). \end{aligned}$$

We recall that in [1] the authors, using extrapolation methods from [16], proved that the operator $G_{B,\Phi}$ generates a strongly continuous semigroup $(T_{B,\Phi}(t))_{t \geq 0}$. In this paper we compute the resolvent of $G_{B,\Phi}$ and show that it satisfies the conditions of the Hille–Yosida theorem. This approach allows us to obtain information on the robustness of the system under small perturbations of the delay operator Φ . For the concrete examples of delay operators we refer to [6].

Theorem 3.2. Let $e_\lambda : X \rightarrow E$ be the function defined by $[e_\lambda x](t) := e^{\lambda t} U(t, 0)x$ for $t \leq 0$, $x \in X$ and $\operatorname{Re} \lambda > \omega(\mathcal{U})$. Let the constants K and ω be defined as in Proposition 2.8. Then the following assertions hold:

(i) The set $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > K\|\Phi\| + \omega\} \subset \rho(G_{B,\Phi})$ and for $\operatorname{Re} \lambda > K\|\Phi\| + \omega$ the resolvent of $G_{B,\Phi}$ satisfies

$$R(\lambda, G_{B,\Phi})f = e_\lambda R(\lambda, B)\Phi R(\lambda, G_{B,0})f + R(\lambda, G_{B,0})f, \quad f \in E. \quad (3.1)$$

(ii) $\|R(\lambda, G_{B,\Phi})\| \leq \frac{K}{(\operatorname{Re} \lambda - K\|\Phi\| - \omega)}$ for $\operatorname{Re} \lambda > K\|\Phi\| + \omega$.

(iii) For $\operatorname{Re} \lambda > K^2 \|\Phi\| + \omega$ we have

$$\|R(\lambda, G_{B,\Phi})^n\| \leq \frac{K}{(\operatorname{Re} \lambda - K^2 \|\Phi\| - \omega)^n} \quad \text{for all } n \in \mathbb{N}. \quad (3.2)$$

Proof. (i) Note that, for $\lambda > K \|\Phi\| + \omega$, the equation

$$\begin{aligned} u(t) &= e^{\lambda t} U(t, 0) R(\lambda, B) (f(0) + \Phi u) \\ &\quad + \int_t^0 e^{\lambda(t-\xi)} U(t, \xi) f(\xi) d\xi \quad \text{for } t \leq 0 \end{aligned} \quad (3.3)$$

is equivalent to

$$u = e_\lambda R(\lambda, B) \Phi u + R(\lambda, G_{B,0}) f. \quad (3.4)$$

If for each $f \in E$ and $\operatorname{Re} \lambda > K \|\Phi\| + \omega$ this equation has a unique solution $u \in E$, then $u(0) = R(\lambda, B) (f(0) + \Phi u)$. This is equivalent to

$$(\lambda - B)u(0) = [(\lambda - G)u](0) + \Phi u \quad \text{or} \quad [Gu](0) = Bu(0) + \Phi u.$$

Hence, by Lemma 2.5, $u \in D(G_{B,\Phi})$ and $u = R(\lambda, G_{B,\Phi}) f$. Therefore, to prove (i) we have to verify that, for each $f \in E$ and $\operatorname{Re} \lambda > K \|\Phi\| + \omega$, Eq. (3.4) has a unique solution $u \in E$. Let $M_\lambda : E \rightarrow E$ be the linear operator defined as $M_\lambda := e_\lambda R(\lambda, B) \Phi$.

Since λ satisfies $\operatorname{Re} \lambda > K \|\Phi\| + \omega$, we have that M_λ is bounded with

$$\|M_\lambda\| \leq \frac{K \|\Phi\|}{\operatorname{Re} \lambda - \omega} < 1.$$

Therefore, the operator $I - M_\lambda$ is invertible, and Eq. (3.4) has a unique solution $u = (I - M_\lambda)^{-1} R(\lambda, G_{B,0}) f$. Thus,

$$R(\lambda, G_{B,\Phi}) f = M_\lambda R(\lambda, G_{B,\Phi}) f + R(\lambda, G_{B,0}) f.$$

(ii) By the Neumann series $(I - M_\lambda)^{-1} = \sum_{n=0}^{\infty} M_\lambda^n$ we have that, for $\operatorname{Re} \lambda > K \|\Phi\| + \omega$,

$$\begin{aligned} \|R(\lambda, G_{B,\Phi})\| &= \left\| \sum_{n=0}^{\infty} M_\lambda^n R(\lambda, G_{B,0}) \right\| \leq \frac{K}{(\operatorname{Re} \lambda - \omega)} \sum_{n=0}^{\infty} \|M_\lambda^n\| \\ &\leq \frac{K}{(\operatorname{Re} \lambda - \omega)} \sum_{n=0}^{\infty} \left(\frac{K \|\Phi\|}{\operatorname{Re} \lambda - \omega} \right)^n = \frac{K}{(\operatorname{Re} \lambda - K \|\Phi\| - \omega)}. \end{aligned}$$

(iii) We shall prove this by induction. By (3.1) we obtain that

$$\begin{aligned} R(\lambda, G_{B,\Phi})^n &= e_\lambda R(\lambda, B) \Phi R(\lambda, G_{B,\Phi})^n + R(\lambda, G_{B,0}) R(\lambda, G_{B,\Phi})^{n-1} \\ &= e_\lambda R(\lambda, B) \Phi R(\lambda, G_{B,\Phi})^n \\ &\quad + R(\lambda, G_{B,0}) e_\lambda R(\lambda, B) \Phi R(\lambda, G_{B,\Phi})^{n-1} \\ &\quad + R(\lambda, G_{B,0})^2 R(\lambda, G_{B,\Phi})^{n-2} \\ &= \dots \end{aligned}$$

$$\begin{aligned}
&= e_\lambda R(\lambda, B) \Phi R(\lambda, G_{B,\Phi})^n \\
&\quad + R(\lambda, G_{B,0}) e_\lambda R(\lambda, B) \Phi R(\lambda, G_{B,\Phi})^{n-1} \\
&\quad + R(\lambda, G_{B,0})^2 e_\lambda R(\lambda, B) \Phi R(\lambda, G_{B,\Phi})^{n-2} + \cdots \\
&\quad + R(\lambda, G_{B,0})^n.
\end{aligned} \tag{3.5}$$

Clearly, (3.2) holds for $n = 1$. If it holds for $n - 1$, we prove it for n .

In fact, for $\operatorname{Re} \lambda > K^2 \|\Phi\| + \omega$, we obtain, by (3.5) and induction hypothesis, that

$$\begin{aligned}
\|R(\lambda, G_{B,\Phi})^n\| &\leq \frac{K^2 \|\Phi\|}{\operatorname{Re} \lambda - \omega} \|R(\lambda, G_{B,\Phi})^n\| \\
&\quad + \frac{K^3 \|\Phi\|}{(\operatorname{Re} \lambda - \omega)^2 (\operatorname{Re} \lambda - \omega - K^2 \|\Phi\|)^{n-1}} \\
&\quad + \frac{K^3 \|\Phi\|}{(\operatorname{Re} \lambda - \omega)^3 (\operatorname{Re} \lambda - \omega - K^2 \|\Phi\|)^{n-2}} + \cdots \\
&\quad + \frac{K^3 \|\Phi\|}{(\operatorname{Re} \lambda - \omega)^n (\operatorname{Re} \lambda - \omega - K^2 \|\Phi\|)} + \frac{K}{(\operatorname{Re} \lambda - \omega)^n}.
\end{aligned}$$

Putting $a := \operatorname{Re} \lambda - \omega$, $b := \operatorname{Re} \lambda - \omega - K^2 \|\Phi\|$, this yields

$$\begin{aligned}
\frac{b}{a} \|R(\lambda, G_{B,\Phi})^n\| &\leq K \left[\frac{K^2 \|\Phi\|}{a^2 b} \left(\frac{1}{b^{n-2}} + \frac{1}{ab^{n-3}} + \cdots + \frac{1}{a^{n-2}} \right) + \frac{1}{a^n} \right] \\
&= K \left[\frac{K^2 \|\Phi\|}{a^2 b} \left(\frac{\frac{1}{a^{n-1}} - \frac{1}{b^{n-1}}}{\frac{1}{a} - \frac{1}{b}} \right) + \frac{1}{a^n} \right] \\
&= \frac{K}{ab^{n-1}} \quad (\text{note that } a - b = K^2 \|\Phi\|).
\end{aligned}$$

Hence,

$$\|R(\lambda, G_{B,\Phi})^n\| \leq \frac{K}{b^n} = \frac{K}{(\operatorname{Re} \lambda - \omega - K^2 \|\Phi\|)^n}. \quad \square$$

Since $G_{B,\Phi}$ is densely defined, we obtain the following results.

Corollary 3.3. *The operator $G_{B,\Phi}$ generates a strongly continuous semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ satisfying*

$$\|T_{B,\Phi}(t)\| \leq K e^{(K^2 \|\Phi\| + \omega)t},$$

where the constants K and ω are defined as in Proposition 2.8.

Corollary 3.4. *If the backward evolution family \mathcal{U} and the semigroup $(e^{tB})_{t \geq 0}$ are exponentially stable and $\|\Phi\|$ is small enough, then the semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ is also exponentially stable.*

Proof. The assumption that \mathcal{U} and $(e^{tB})_{t \geq 0}$ are exponentially stable means that $\omega = \max\{\omega_1, \omega_2\} < 0$. Therefore, if $\|\Phi\| < -\omega/K^2$, then the semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ is also exponentially stable. \square

In the following example we shall determine the “sufficient smallness” of $\|\Phi\|$ more explicitly.

Example 3.5. Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary. The Dirichlet Laplacian generates an analytic semigroup $(e^{t\Delta})_{t \geq 0}$ on $X := L^2(\Omega)$. We then take operators $A(s)$ as

$$A(s) := a(s)\Delta,$$

where the function $a(\cdot) \in L^1_{\text{loc}}(\mathbb{R}_-)$ satisfies $a(\cdot) \geq \gamma > 0$ for some constant γ . These operators generate a backward evolution family $(U(r, s))_{r \leq s \leq 0}$ given by

$$U(r, s) = e^{(\int_r^s a(\tau) d\tau)\Delta} \quad \text{for } r \leq s \leq 0.$$

We then have

$$\|U(r, s)\| = e^{(\int_r^s a(\tau) d\tau)\lambda_0} \leq e^{\gamma\lambda_0(s-r)} \quad \text{for } r \leq s \leq 0,$$

where $\lambda_0 < 0$ denotes the largest eigenvalue of Δ . Therefore, we can choose in Definition 2.1 the constants $N = 1$ and $\omega_1 = \gamma\lambda_0 < 0$. We now define the delay operator Φ by

$$\Phi f := \int_{-\infty}^0 \varphi(s) f(s) ds \quad \text{for } f \in E,$$

where $\varphi(\cdot) \in L^1(\mathbb{R})$. We then have

$$\|\Phi\| \leq \|\varphi(\cdot)\|_{L^1}.$$

Let now B generate a semigroup $(e^{tB})_{t \geq 0}$ satisfying $\|e^{tB}\| \leq Me^{\omega_2 t}$ with $\omega_2 < 0$. From the definition of $(T_{B,0}(t))_{t \geq 0}$ we obtain

$$\|T_{B,0}(t)\| \leq Me^{\max\{\gamma\lambda_0, \omega_2\}t}, \quad t \geq 0.$$

Hence, in Corollary 3.3 we can choose $K = M$. Therefore, if

$$\|\varphi(\cdot)\|_{L^1} < -\frac{\max\{\gamma\lambda_0, \omega_2\}}{M^2},$$

then the semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ is exponentially stable.

4. Spectra and hyperbolicity of evolution semigroups

In this section we first compute the spectra of the evolution semigroup $T_{B,0}(t)$ and its generator. This will be used to prove the robustness of the hyperbolicity of the semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ under small perturbations by the delay operator Φ . We first compare $T_{B,0}(t)$

to its restriction to the subspace $C_{00} := \{f \in E: f(0) = 0\}$. This restriction has already been studied in [11,18].

Lemma 4.1. Denote by $(T_0(t))_{t \geq 0}$ the restriction of $(T_{B,0}(t))_{t \geq 0}$ to the space C_{00} and let G_0 be its generator. Then the following assertions hold:

$$(i) \quad \sigma(T_{B,0}(t)) \subseteq \sigma(T_0(t)) \cup \sigma(e^{tB}) \quad \text{for } t \geq 0. \quad (4.1)$$

$$(ii) \quad \sigma(G_{B,0}) \cup \sigma(B) = \sigma(G_0) \cup \sigma(B). \quad (4.2)$$

Proof. (i) Endow $X \oplus C_{00}$ with the 1-norm

$$\|(x, f)\| := \|f\| + \|x\| \quad \text{for } (x, f) \in X \oplus C_{00}.$$

For a fixed continuous real valued function φ with compact support satisfying $\varphi(0) = 1$, we consider the linear operator

$$\mathcal{J} : E \rightarrow X \oplus C_{00}, \quad f \mapsto (f(0), f - \varphi(\cdot)f(0)).$$

Then \mathcal{J} is an isomorphism and its inverse is given by

$$\mathcal{J}^{-1} : X \oplus C_{00} \rightarrow E, \quad (x, f) \mapsto f + \varphi(\cdot)x.$$

Therefore, by similarity, the operators

$$\widehat{T}(t) := \mathcal{J}T_{B,0}(t)\mathcal{J}^{-1} = \begin{pmatrix} e^{tB} & 0 \\ (T_{B,0}(t) - e^{tB})\varphi(\cdot) & T_0(t) \end{pmatrix}$$

form a semigroup satisfying $\sigma(\widehat{T}(t)) = \sigma(T_{B,0}(t))$. Let now $\lambda \in \rho(T_0(t)) \cap \rho(e^{tB})$. Then the operator

$$\begin{pmatrix} \lambda - e^{tB} & 0 \\ (T_{B,0}(t) - e^{tB})\varphi(\cdot) & \lambda - T_0(t) \end{pmatrix}$$

is invertible with inverse

$$\begin{pmatrix} (\lambda - e^{tB})^{-1} & 0 \\ -(\lambda - T_0(t))^{-1}[(T_{B,0}(t) - e^{tB})\varphi(\cdot)](\lambda - e^{tB})^{-1} & (\lambda - T_0(t))^{-1} \end{pmatrix}.$$

Hence $\lambda \in \rho(\widehat{T}(t)) = \rho(T_{B,0}(t))$. This means that $\rho(T_0(t)) \cap \rho(e^{tB}) \subseteq \rho(T_{B,0}(t))$. Thus, (i) follows.

(ii) By Proposition 2.8, we have $\rho(G_0) \cap \rho(B) \subseteq \rho(G_{B,0})$. Hence,

$$\sigma(G_{B,0}) \subseteq \sigma(G_0) \cup \sigma(B). \quad (4.3)$$

It remains to prove that

$$\sigma(G_0) \subseteq \sigma(G_{B,0}) \cup \sigma(B). \quad (4.4)$$

In fact, if $\lambda - G_{B,0}$ is injective, then so is $\lambda - G_0$ because G_0 is the restriction of $G_{B,0}$ to C_{00} .

Let now $\lambda \in \rho(B)$ and $\lambda - G_{B,0}$ be surjective. We will verify that $\lambda - G_0$ is also surjective. Indeed, let $f \in C_{00}$ be arbitrary. Then, by the surjectivity of $\lambda - G_{B,0}$, there exists a function $u \in D(G_{B,0})$ such that $(\lambda - G_{B,0})u = f$. By definition of $G_{B,0}$ we have that

$0 = f(0) = \lambda u(0) - [G_{B,0}u](0) = (\lambda - B)u(0)$. Therefore, $u(0) = 0$ and $u \in C_{00}$. Hence, $(\lambda - G_0)u = (\lambda - G_{B,0})u = f$. Thus, $\lambda - G_0$ is surjective. This yields

$$\rho(G_{B,0}) \cap \rho(B) \subseteq \rho(G_0),$$

and inclusion (4.4) follows. \square

In [11, Corollary 2.4] it has been proved that a spectral mapping theorem holds for the semigroup $(T_0(t))_{t \geq 0}$. More precisely, we have

$$\sigma(G_0) = \{\lambda \in \mathbb{C}: \operatorname{Re} \lambda \leq \omega(\mathcal{U})\}$$

and

$$\sigma(T_0(t)) \setminus \{0\} = e^{t\sigma(G_0)}, \quad t > 0. \quad (4.5)$$

By this and Lemma 4.1 we obtain the following.

Theorem 4.2. *The spectral equality*

$$[\sigma(T_{B,0}(t)) \cup \sigma(e^{tB})] \setminus \{0\} = [e^{t\sigma(G_0)} \cup \sigma(e^{tB})] \setminus \{0\}, \quad t \geq 0, \quad (4.6)$$

holds with the operator G_0 as in Lemma 4.1.

Proof. By Lemma 4.1 and (4.5) we have that

$$\begin{aligned} [\sigma(T_{B,0}(t)) \cup \sigma(e^{tB})] \setminus \{0\} &\stackrel{\text{by (4.1)}}{\subseteq} [\sigma(T_0(t)) \cup \sigma(e^{tB})] \setminus \{0\} \\ &\stackrel{\text{by (4.5)}}{=} [e^{t\sigma(G_0)} \cup \sigma(e^{tB})] \setminus \{0\} \\ &= [e^{t\sigma(G_0)} \cup e^{t\sigma(B)} \cup \sigma(e^{tB})] \setminus \{0\} \\ &= [e^{t(\sigma(G_0) \cup \sigma(B))} \cup \sigma(e^{tB})] \setminus \{0\} \\ &\stackrel{\text{by (4.2)}}{=} [e^{t(\sigma(G_{B,0}) \cup \sigma(B))} \cup \sigma(e^{tB})] \setminus \{0\} \\ &= [e^{t\sigma(G_{B,0})} \cup e^{t\sigma(B)} \cup \sigma(e^{tB})] \setminus \{0\} \\ &\subseteq [\sigma(T_{B,0}(t)) \cup \sigma(e^{tB})] \setminus \{0\}. \end{aligned}$$

Thus, (4.6) follows. \square

Using the spectral characterization of hyperbolic semigroups (see [3, Theorem V.1.15]), the above theorem allows the following consequence.

Corollary 4.3. *If the operator $(B, D(B))$ generates a hyperbolic semigroup $(e^{tB})_{t \geq 0}$ and if the backward evolution family \mathcal{U} is exponentially stable, then the semigroup $(T_{B,0}(t))_{t \geq 0}$ is hyperbolic.*

Proof. The assumption that \mathcal{U} is exponentially stable implies that $\omega(\mathcal{U}) < 0$, hence $s(G_0) < 0$ by (4.5). Therefore, $\sigma(G_0) \cap i\mathbb{R} = \emptyset$. By the hyperbolicity of $(e^{tB})_{t \geq 0}$ we have

$$(e^{t\sigma(G_0)} \cup \sigma(e^{tB})) \cap e^{i\mathbb{R}} = \emptyset.$$

The hyperbolicity of $(T_{B,0}(t))_{t \geq 0}$ now follows from (4.6) and [3, Theorem V.1.15]. \square

The main purpose of this section is to prove the robustness of hyperbolicity of the semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ under small perturbations of the delay operator Φ . To do so we need the following characterization of the hyperbolicity of a semigroup (see [14, Theorem 2.6.2]).

Theorem 4.4. *Let $(T(t))_{t \geq 0}$ be a C_0 semigroup on Banach space X and A be its generator. Then the following assertions are equivalent:*

- (i) $(T(t))_{t \geq 0}$ is hyperbolic.
- (ii) $i\mathbb{R} \subset \rho(A)$ and

$$(C, 1) \sum_{k \in \mathbb{Z}} R(i\omega + ik, A)x := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n R(i\omega + ik, A)x$$

converges for all $\omega \in \mathbb{R}$ and $x \in X$.

We note that the above theorem is taken from [14, Theorem 2.6.2], while its proof is essentially due to Greiner and Schwarz [5, Theorem 1.1 and Corollary 1.2]. A continuous version of the above theorem is proved by Kaashoek and Verduyn Lunel in [8, Theorem 4.1].

In order to apply this theorem we have to compute the resolvent $R(\lambda, G_{B,\Phi})$ from the resolvent $R(\lambda, G_{B,0})$. This can be done as follows.

Lemma 4.5. *Let the backward evolution \mathcal{U} be exponentially stable and the operator $(B, D(B))$ be the generator of a hyperbolic semigroup $(e^{tB})_{t \geq 0}$. Then for sufficiently small $\|\Phi\|$ there exist an open strip Σ containing the imaginary axis and a function H_λ which is analytic and uniformly bounded on Σ such that*

$$R(\lambda, G_{B,\Phi}) = H_\lambda R(\lambda, G_{B,0}) \quad \text{for } \lambda \in \Sigma. \quad (4.7)$$

Proof. By [8, Theorem 4.1] and the hyperbolicity of $(e^{tB})_{t \geq 0}$, we obtain that there exist constants $P_1, \nu > 0$ such that

$$\|R(\lambda, B)\| \leq P_1 \quad \text{for all } |\operatorname{Re} \lambda| < \nu.$$

By the exponential stability of \mathcal{U} , there exist constants $\omega_1 > 0$ and K_1 such that

$$\|U(t, s)\| < K_1 e^{-\omega_1(s-t)} \quad \text{for all } t \leq s \leq 0. \quad (4.8)$$

Let now ω be a real number such that $0 < \omega < \min\{\omega_1, \nu\}$. We then put

$$\Sigma := \{\lambda \in \mathbb{C}: |\operatorname{Re} \lambda| < \omega\}$$

and

$$P := \sup_{\lambda \in \Sigma} \|R(\lambda, B)\|. \quad (4.9)$$

As in the proof of Theorem 3.2, we first verify that for each $f \in E$ and $\lambda \in \Sigma$ Eq. (3.4) has a unique solution $u \in E$.

Let $M_\lambda : E \rightarrow E$ be the linear operator defined as $M_\lambda := e_\lambda R(\lambda, B)\Phi$ with e_λ as in Theorem 3.2. For $\lambda \in \Sigma$, this operator is bounded and satisfies

$$\|M_\lambda\| \leq K_1 P \|\Phi\| < 1$$

if, in addition,

$$\|\Phi\| < \frac{1}{K_1 P}.$$

Therefore, the operator $I - M_\lambda$ is invertible, and Eq. (3.4) has a unique solution $u = (I - M_\lambda)^{-1} R(\lambda, G_{B,0})f$. Putting $H_\lambda := (I - M_\lambda)^{-1}$ yields

$$R(\lambda, G_{B,\Phi}) = H_\lambda R(\lambda, G_{B,0}).$$

Moreover, the Neumann series yields

$$H_\lambda = (I - M_\lambda)^{-1} = \sum_{n=0}^{\infty} M_\lambda^n, \quad (4.10)$$

hence

$$\|H_\lambda\| \leq \sum_{n=0}^{\infty} \|M_\lambda\|^n \leq \sum_{n=0}^{\infty} (K_1 P \|\Phi\|)^n = \frac{1}{1 - K_1 P \|\Phi\|}$$

for all $\lambda \in \Sigma$ and $\|\Phi\| < 1/(K_1 P)$. The analyticity of H_λ follows from that of M_λ and the uniform convergence of the Neumann series (4.10) for all $\lambda \in \Sigma$. \square

We now come to our main result about exponential dichotomy of solutions of Eqs. (1.1) and (1.2).

Theorem 4.6. *Let the backward evolution \mathcal{U} be exponentially stable and the operator $(B, D(B))$ be generator of a hyperbolic semigroup $(e^{tB})_{t \geq 0}$. Then, for sufficiently small $\|\Phi\|$, the semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ is hyperbolic.*

Proof. By Corollary 4.3, the evolution semigroup $(T_{B,0}(t))_{t \geq 0}$ is hyperbolic. We first prove that, for sufficiently small $\|\Phi\|$, the sum $(1/N) \sum_{n=0}^{N-1} \sum_{k=-n}^n R(i\omega + ik, G_{B,\Phi})$ is bounded in $\mathcal{L}(E)$. In fact, by Lemma 4.5, we have

$$\begin{aligned} & \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n [R(i\omega + ik, G_{B,\Phi})f](s) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n [(1 + M_{i\omega+ik} + M_{i\omega+ik}^2 + \cdots)R(i\omega + ik, G_{B,0})f](s) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n [R(i\omega + ik, G_{B,0})f](s) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n e^{(i\omega+ik)s} U(s, 0) R(i\omega + ik, B) \Phi R(i\omega + ik, G_{B,0}) f \\
& + \dots
\end{aligned} \tag{4.11}$$

for $s \in \mathbb{R}_-$.

Note that the semigroup $(T_{B,0}(t))_{t \geq 0}$ is hyperbolic, hence $e^{-2\pi i\omega} \in \rho(T_{B,0}(2\pi))$ for all $\omega \in \mathbb{R}$. Using the formula (see [3, Lemma II.1.9])

$$R(\lambda, G_{B,0})(1 - e^{-\lambda t} T_{B,0}(t)) = \int_0^t e^{-\lambda s} T_{B,0}(s) ds \quad \text{for } \lambda \in \rho(G_{B,0}),$$

we obtain

$$R(i\omega + ik, G_{B,0}) = \int_0^{2\pi} e^{-(i\omega+ik)t} T_{B,0}(t) (1 - e^{-2\pi i\omega} T_{B,0}(2\pi))^{-1} dt.$$

The first term of (4.11) can now be computed as

$$\begin{aligned}
& \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n R(i\omega + ik, G_{B,0}) f \\
& = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n \int_0^{2\pi} e^{-(i\omega+ik)t} T_{B,0}(t) (1 - e^{-2\pi i\omega} T_{B,0}(2\pi))^{-1} f dt \\
& = \int_0^{2\pi} \left[\frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n e^{-ikt} \right] e^{-i\omega t} T_{B,0}(t) (1 - e^{-2\pi i\omega} T_{B,0}(2\pi))^{-1} f dt \\
& = \int_0^{2\pi} \sigma_N(t) e^{-i\omega t} T_{B,0}(t) (1 - e^{-2\pi i\omega} T_{B,0}(2\pi))^{-1} f dt.
\end{aligned}$$

Here, $\sigma_N(t) := (1/N) \sum_{n=0}^{N-1} \sum_{k=-n}^n e^{-ikt}$. Since

$$\sigma_N(t) = \frac{1 - \cos(Nt)}{N(1 - \cos t)} \geq 0 \quad \text{and} \quad \int_0^{2\pi} \sigma_N(t) dt = 2\pi \tag{4.12}$$

(see [5, Theorem 1.1]), the norm of the first term in (4.11) can be estimated by

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n R(i\omega + ik, G_{B,0}) f \right\| \leq C_1 \|f\| \tag{4.13}$$

with $C_1 := 2\pi \sup_{0 \leq w \leq 1} \{ \| (1 - e^{-2\pi i\omega} T_{B,0}(2\pi))^{-1} \| \} \sup_{0 \leq t \leq 2\pi} \{ \| T_{B,0}(t) \| \}$.

We now compute the second term of (4.11). For $s \in \mathbb{R}_-$, we have

$$\begin{aligned}
 & \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n M_{i\omega+ik} R(i\omega+ik, G_{B,0}) f(s) \\
 &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n e^{(i\omega+ik)s} U(s, 0) R(i\omega+ik, B) \Phi R(i\omega+ik, G_{B,0}) f \\
 &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n e^{(i\omega+ik)s} U(s, 0) \int_0^{2\pi} e^{-(i\omega+ik)\tau} e^{\tau B} (1 - e^{2\pi B})^{-1} d\tau \\
 &\quad \times \Phi \int_0^{2\pi} e^{-(i\omega+ik)t} T_{B,0}(t) (1 - e^{-2\pi i\omega} T_{B,0}(2\pi))^{-1} f dt \\
 &= \int_0^{2\pi} \int_0^{2\pi} \left[\frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n e^{-ik(t+\tau-s)} \right] e^{-i\omega(t+\tau-s)} U(s, 0) e^{\tau B} (1 - e^{2\pi B})^{-1} \\
 &\quad \times \Phi T_{B,0}(t) (1 - e^{-2\pi i\omega} T_{B,0}(2\pi))^{-1} f d\tau dt \\
 &= \int_0^{2\pi} \int_0^{2\pi} \sigma_N(t+\tau-s) e^{-i\omega(t+\tau-s)} U(s, 0) e^{\tau B} (1 - e^{2\pi B})^{-1} \\
 &\quad \times \Phi T_{B,0}(t) (1 - e^{-2\pi i\omega} T_{B,0}(2\pi))^{-1} f d\tau dt.
 \end{aligned}$$

Therefore, using (4.8) and (4.12), the norm of the second term of (4.11) can be estimated by

$$C_1 K_1 C_2 \|\Phi\| \|f\| \quad \text{with } C_2 := 2\pi \|(1 - e^{2\pi B})^{-1}\| \sup_{0 \leq t \leq 2\pi} \{ \|e^{tB}\| \} \quad (4.14)$$

and K_1, C_1 as in (4.8), (4.13), respectively.

By induction, the norm of the n th term of (4.11) is estimated by

$$C_1 (K_1 C_2 \|\Phi\|)^n \|f\|.$$

Moreover, the series $\sum_{n=0}^{\infty} C_1 (K_1 C_2 \|\Phi\|)^n$ converges if $\|\Phi\| < 1/(K_1 C_2)$. Hence, for these $\|\Phi\|$ the sum $(1/N) \sum_{n=0}^{N-1} \sum_{k=-n}^n R(i\omega+ik, G_{B,\Phi})$ is bounded in $\mathcal{L}(E)$.

We now prove the convergence of $(C, 1) \sum_{k \in \mathbb{Z}} R(i\omega+ik, G_{B,\Phi}) f$ for $\omega \in \mathbb{R}$ and $f \in E$. This can be done by using the idea from [5, Theorem 1.1]. By [17, III.4.5], it is sufficient to show convergence on a dense subset. From $i\mathbb{R} \subset \rho(G_{B,\Phi})$ and the spectral mapping theorem for the residual spectrum (see [3, Theorem IV.3.7]) we obtain that $e^{-2\pi i\omega}$ does not belong to the residual spectrum $R\sigma(T_{B,\Phi}(2\pi))$. This implies that $(1 - e^{-2\pi i\omega} T_{B,\Phi}(2\pi))E$ is a dense subset of E . Let $f := (1 - e^{-2\pi i\omega} T_{B,\Phi}(2\pi))g$. Then

$$\frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n R(i\omega+ik, G_{B,\Phi}) (1 - e^{-2\pi i\omega} T_{B,\Phi}(2\pi))g$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^n \int_0^{2\pi} e^{-(i\omega+ik)s} T_{B,\Phi}(s) g \, ds. \quad (4.15)$$

Now $e^{-i\omega \cdot} T_{B,\Phi}(\cdot)g$ is a continuous function with Fourier coefficients

$$Q_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-(i\omega+ik)s} T_{B,\Phi}(s) g \, ds.$$

Therefore, by Fejer's theorem, the sum in (4.15) converges as $N \rightarrow \infty$. The assertion of the theorem now follows from Theorem 4.4. \square

The “sufficient smallness” of Φ is computed in the following example.

Example 4.7. We consider again Example 3.5 with the same backward evolution family $U(r, s) := e^{(\int_r^s a(\tau) d\tau)\Delta}$ and the same delay operator $\Phi f := \int_{-\infty}^0 \varphi(s) f(s) \, ds$. However, let now B generate a hyperbolic semigroup $(e^{tB})_{t \geq 0}$ satisfying $\|R(\lambda, B)\| \leq P_1$ for $|\operatorname{Re} \lambda| < \omega_2$ (for instance, we can take B to be a sectorial operator satisfying $\sigma(B) \cap i\mathbb{R} = \emptyset$ as in [9, Example 2.1.4] or [19, Example 4.2]). Take $0 < \omega < \min\{-\gamma\lambda_0, \omega_2\}$ and put

$$\Sigma := \{\lambda \in \mathbb{C}: |\operatorname{Re} \lambda| < \omega\}$$

and

$$P := \max \left\{ \sup_{\lambda \in \Sigma} \{ \|R(\lambda, B)\| \}, 2\pi \| (1 - e^{2\pi B})^{-1} \| \sup_{0 \leq t \leq 2\pi} \{ \|e^{tB}\| \} \right\}.$$

We obtain that the semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ is hyperbolic if

$$\|\varphi(\cdot)\|_{L^1} < \frac{1}{P}.$$

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